Theoretical Computer Science 405 (2008) 101–115

Contents lists available at ScienceDirect

Theoretical Computer Science

journal homepage: www.elsevier.com/locate/tcs

The Zariski spectrum as a formal geometry☆

Peter Schuster

Mathematisches Institut, Universität München, Theresienstraße 39, 80333 München, Germany

ARTICLE INFO

Keywords:
Zariski spectrum
Structure sheaf
Local ring
Universal property
Formal topology
Point-free
Constructive
Predicative

ABSTRACT

We choose formal topology to deal in a basic manner with the Zariski spectra of commutative rings and their structure sheaves. By casting prime and maximal ideals in a secondary role, we thus wish to prepare a constructive and predicative framework for abstract algebraic geometry.

In contrast to the classical approach, neither points nor stalks need occur, let alone any instance of the axiom of choice. As compared with the topos-theoretic treatments that may be rendered predicative as well, the road we follow is built from more elementary material.

The formal counterpart of the structure sheaf which we present first is our guiding example for a notion of a sheaf on a formal topology. We next define the category of formal geometries, a natural abstraction from that of locally ringed spaces. This allows us to eventually phrase and prove, still within the language of opens and sections, the universal property of the Zariski spectrum. Our version appears to be the only one that is explicitly point-free.

© 2008 Elsevier B.V. All rights reserved.

1. Introduction

The representation problem "for every commutative ring A with unit, find a topological space and a sheaf of local rings on it such that A is the ring of global sections of this sheaf" was solved by equipping the prime spectrum of A with the Zariski topology, and this with the so-called structure sheaf. In [41] we extended a certain point-free approach [31,42] to the topological part of this construct: the approach carried out within the constructive and predicative setting of formal topology [34,35,37]. Among other things done in [41], to which we refer also for motivation and background, we proposed some positivity structures for the formal variant of the Zariski topology, and studied the formal points thereof, which study was deepened in [15].

The aim of the present work, which – as [41] – has emerged from [40], is to complete this point-free picture. After providing some material on rings of fractions (Section 2), we transfer the structure sheaf to the setting of formal topology (Section 3), and thus put forward a concept of a sheaf on a formal topology in general. This allows for a representation of each commutative ring by a sheaf of local rings on its formal Zariski topology (Section 4), and for an adaptation to the formal-topological setting of the well-known universal property characteristic of the Zariski spectrum (Section 5).

The main work is to abstract the right category, that of formal geometries, from the one of locally ringed – or geometric – spaces. More specifically, we define a morphism of locally ringed spaces in the realm of formal topology and thus, in particular, without any universal quantification over points. Although the latter objective was met earlier in a topos-theoretic approach [20, 6.51], it was neglected in later locale-theoretic treatments such as [22, V.3.5], where points are used in the proof of the universal property. We now show that points are dispensable for proving this property; our approach appears to be the only one whose point-free character does not require further justification.

☆ This article emerged from the second part of [P. Schuster, Spectra and sheaves in formal topology, Habilitation Thesis, University of Munich, 2003], whose first part prompted [P. Schuster, Formal Zariski topology: Positivity and points, Ann. Pure Appl. Logic, 137 (2006), 317–359].

E-mail address: Peter.Schuster@mathematik.uni-muenchen.de.

0304-3975/$ – see front matter © 2008 Elsevier B.V. All rights reserved.
doi:10.1016/j.tcs.2008.06.030

A predicative proof of the universal property might also be obtained from its early treatment in topos theory [18]. Later, in fact, this was described in terms of geometric theories [44], by which any use of the subobject classifier could be circumvented. (Roughly speaking, geometric logic is tied together with the idea of local truth, and thus supports reasoning with opens rather than points.) As compared with this possible route, our approach to the universal property through formal topology appears to be more direct and elementary.

Note in this context that the existence of a prime ideal in an arbitrary nontrivial commutative ring is equivalent to the Boolean ultrafilter theorem [7]; accordingly, the Zariski spectrum may not have any point at all, so within a general topos that lacks this variant of the axiom of choice [21, p. 258]. Within any predicative framework, on the other hand, the prime spectrum may lack the quality of being a set: as subsets of the given set, the commutative ring A under consideration, the prime ideals are elements of the power class of A, and fail to form a set in general.

While formal topology was originally intended to be expressed in Martin–Löf's intuitionistic type theory (ITT) [27], we keep to Aczel's constructive Zermelo–Fraenkel set theory (CZF) [6] without providing every detail. It should therefore be understood as in CZF when we speak of subsets, quotients, equalisers, functors, and of other concepts that may require a specific treatment in ITT (see, e.g., [11,25,26,37,39]). Unlike ITT, in particular, every subset is meant to be a set in CZF. It is in order to remember that CZF can be interpreted in ITT [1–3], and that a considerable amount of formal topology has already been done in CZF [4,5].

We require acquaintance with [41], *whose conventions and notations we adopt*. Familiarity with [41] is particularly needed wherever we mention the concepts of inequality as possibly stronger than the negation of equality, and of positivity as the dual notion of a covering. However, none of them is necessary for achieving our final goal, the universal property of the formal Zariski topology: the only inequality that occurs in Section 5 is the negation of equality, and from 5.3 there is no talk of positivity. But why do we then take these two concepts into account? Inequality and positivity are present in the literature on constructive algebra [9,10,29,33,43] and formal topology [14,28,35], respectively, and we want to know how they interact with the notions of a ring of fractions and of a sheaf on a formal topology.

2. Rings of fractions

In this section we recall the concept of a ring of fractions, and endow it with an inequality. The properties we list in the sequel are all straightforward to prove in a constructive way, often along classical lines; whence we may leave this task to the reader. Every commutative ring A is assumed to have a unit, and to be a set.

Let S be a multiplicative subset of the commutative ring A. The ring of fractions

$$S^{-1}A = \left\{ \frac{x}{s} : x \in A,\ s \in S \right\}$$

with denominators in S comes with the equality

$$\frac{x}{s} = \frac{y}{t} \iff \exists z \in S\ (zxt = zys)\,,$$

and inherits from A the structure of a commutative ring with zero $\frac{0}{1}$ and unit $\frac{1}{1}$. Sum, additive inverse, and product of fractions are carried over from A to $S^{-1}A$ in the same way in which the operations on the rationals are defined from the ones on the integers (this is the specific case $A = \mathbb{Z}$, $S = \mathbb{Z} \setminus \{0\}$, $S^{-1}A = \mathbb{Q}$ of a ring of fractions). For arbitrary S and A, one can formally define $S^{-1}A$ as the set $A \times S$, with each fraction $\frac{x}{s}$ denoting the ordered pair (x, s). We further endow $S^{-1}A$ with the inequality

$$\frac{x}{s} \neq \frac{y}{t} \iff \forall z \in S\ (zxt \neq zys)\,,$$

which is a ring inequality whenever so is the inequality on A, whereas the former does not necessarily inherit cotransitivity from the latter.

Note that $\frac{x}{s} = 0$ if and only if $zx = 0$ for some $z \in S$, and $\frac{x}{s} \neq 0$ if and only if $tx \neq 0$ for all $t \in S$. In particular, $\frac{x}{s} = 0$ follows from $x = 0$, and $\frac{x}{s} \neq 0$ implies $x \neq 0$. If A is an integral domain and $S \subset\sim \{0\}$, then $\frac{x}{s} \neq 0$ if and only if $x \neq 0$; if, in addition, the inequality $\neq$ on A is tight, then also $\frac{x}{s} = 0$ if and only if $x = 0$.

The invertible elements of $S^{-1}A$ are precisely the fractions $\frac{x}{s}$ with $yx \in S$ for some y belonging to A but not necessarily to S; in particular, $\frac{x}{s}$ is invertible already if $x \in S$. If A is an integral domain with a ring inequality, and $S \subset\sim \{0\}$, then $S^{-1}A$ is a field. Moreover, $S^{-1}A$ is trivial if and only if $0 \in S$,[1] and if $S^{-1}A$ is nontrivial, then $S \subset\sim \{0\}$. If A is equipped with the denial inequality, then so is $S^{-1}A$; in particular, $S^{-1}A$ is nontrivial precisely when $0 \notin S$.

If S is a filter, then $\frac{x}{s} \in \left(S^{-1}A\right)^*$ is actually equivalent to $x \in S$; whence $S^{-1}A$ is nontrivial precisely when $S \subset\sim \{0\}$. Moreover, S is a prime filter if and only if $S^{-1}A$ is a local ring, and S is a filter with $S \subset \neg\{0\}$. So if S is a filter, then $S^{-1}A$ is a

[1] Richman based a couple of subtle proof techniques upon this characterisation [32], as marginal it may seem at first glance. Following an idea of Setzer, Carlström generalised the concept of a ring of fractions to that of a so-called wheel of fractions [12], for which – in contrast to the former – division by zero does not always render trivial all the fractions under consideration.

nontrivial local ring if and only if S is a prime filter with $S \subset \sim \{0\}$; if, in addition, A comes with the denial inequality, then $S^{-1}A$ is a nontrivial local ring precisely when S is a prime filter.[2]

The canonical ring homomorphism

$$\pi : A \to S^{-1}A,\ x \mapsto \frac{x}{1}$$

maps every element of S to an invertible element of $S^{-1}A$. Moreover, π is universal with respect to this property: if $\psi : A \to B$ is a ring homomorphism such that $\psi(s)$ is invertible in B for every $s \in S$, then there is exactly one ring homomorphism $\Psi : S^{-1}A \to B$ with $\Psi \circ \pi = \psi$ (which is given by $\Psi(\frac{x}{s}) = \psi(x)\psi(s)^{-1}$ for all $x \in A$ and $s \in S$). In particular, π is a – not necessarily surjective – epimorphism, which is an isomorphism precisely when $S \subset A^*$. Note also that π is strongly extensional, and that Ψ is strongly extensional if (and only if) ψ is strongly extensional.

An example of a ring of fractions that will frequently occur is

$$A_a = M(a)^{-1} A,$$

where

$$M(a) = \left\{a^n : n \in \mathbb{N}_0\right\}$$

is the multiplicative submonoid of A generated by $a \in A$. In this case the canonical ring homomorphism will be denoted by

$$\pi_a : A \to A_a,\ x \mapsto \frac{x}{1}.$$

With $M(a)$ at hand, moreover, the radical of the ideal $I(U)$ generated by a subset U of A can be characterised as the subset $R(U)$ of A for which

$$a \in R(U) \iff M(a) \between I(U).$$

Though almost trivial, the following observations will later be of considerable use.

Lemma 1. *If A is a commutative ring and $b \in A$, then*

$$R(b) = \left\{a \in A : \pi_a(b) \in (A_a)^*\right\}.$$

Proof. For every $a \in A$, we have $a \in R(b)$ if and only if $M(a)$ meets $I(b)$, which is to say that $bc \in M(a)$ for some $c \in A$; in other words, $\pi_a(b) \in (A_a)^*$. □

Corollary 2. *Let A be a commutative ring and $a, b \in A$. If $a \in R(b)$, then $\pi_a : A \to A_a$ factors uniquely through $\pi_b : A \to A_b$: that is, there is exactly one ring homomorphism $r_{a,b} : A_b \to A_a$ with $r_{a,b} \circ \pi_b = \pi_a$.*

Proof. If $a \in R(b)$, then $\pi_a(M(b)) \subset (A_a)^*$ by Lemma 1; whence existence and uniqueness of $r_{a,b}$ are ensured by the universal property of π_b. □

Note that A_a is trivial if and only if $0 \in M(a)$. Moreover, if A_a is nontrivial, then $M(a) \subset \sim \{0\}$; the converse holds whenever $\neq$ is the denial inequality.

3. Presheaves and sheaves

Following the well-known definition of a sheaf on a basis of a topology as, for example, given in [24, p. 69], we now adapt the so-called structure sheaf on the spectrum of a ring to the formal setting, and thus propose a notion of a sheaf on an arbitrary formal topology based on a multiplicative monoid. The need to provide the structure sheaf is due to the fact that it reflects the algebraic structure of the commutative ring A, which almost disappears in the transition from A to the classical topological space Spec (A) and still partially during that from A to the formal Zariski topology of A. Unlike the multiplicative monoid of A, which is the same as the monoid underlying the formal Zariski topology of A, the additive structure of A has been folded into covering and positivity. (It is tempting to think, however, that addition might be recovered from covering and positivity.)

For the moment we only expect a formal topology to be based on a set A which is a multiplicative monoid, and to come with a (reflexive and transitive) covering relation $\lhd$. Since later we will be only concerned with sheaves of rings, we follow the time-honoured tradition to begin with presheaves of abelian groups with underlying sets rather than, as common to topos theory, with presheaves of arbitrary sets.

[2] While any ring whatsoever is nontrivial with respect to the natural inequality (for which $x \neq y$ precisely when $x - y$ is invertible), by a nontrivial local ring we understand a local ring that it is nontrivial with respect to the inequality with which it arrives. In particular, we rather follow [29] than the literature on topos theory, and do not include into the definition of a local ring that it is nontrivial with respect to the denial inequality: that is, $\neg(0 \in A^*)$ or, equivalently, $\neg(0 = 1)$.

Definition 1. A *presheaf* on a formal topology is a family $\mathcal{F}$ of abelian groups

$$\mathcal{F}(a) \quad (a \in A)$$

together with a family of group homomorphisms

$$r_{a,b} : \mathcal{F}(b) \to \mathcal{F}(a) \quad (a \lhd b)$$

subject to the conditions

$$r_{a,a} = \mathrm{id}_{\mathcal{F}(a)} \quad \text{and} \quad a \lhd b \lhd c \Longrightarrow r_{a,c} = r_{a,b} \circ r_{b,c}$$

for all $a, b, c \in A$.

In particular, we may use the common notation $s|_a$ for $r_{a,b}(s)$ whenever $s \in \mathcal{F}(b)$ and $a \lhd b$. As usual, we call each $r_{a,b}$ a *restriction mapping*, and each element of $\mathcal{F}(a)$ a *section* of $\mathcal{F}$ over a.

Definition 2. A *sheaf* on a formal topology is a presheaf $\mathcal{F}$ for which

$$\mathcal{F}(a) \underset{i_a}{\to} \prod_{b \in U} \mathcal{F}(ab) \underset{q_a}{\overset{p_a}{\rightrightarrows}} \prod_{(c,d) \in U \times U} \mathcal{F}(acd) \tag{1}$$

is an equaliser diagram[3] for all $a \in A$ and $U \subset A$ with $a \lhd U$, where i, p, q are so that $i_a(s) = (s|_{ab})_{b \in U}$ for $s \in \mathcal{F}(a)$, and

$$p_a\left((t_b)_{b\in U}\right)_{(c,d)} = t_c|_{acd}, \qquad q_a\left((t_b)_{b\in U}\right)_{(c,d)} = t_d|_{acd}$$

for $t_b \in \mathcal{F}(ab)$ with $b \in U$ and $(c, d) \in U \times U$.

It is in order to comment on the occurrence, in Definition 2, of a quantification over the subsets of A, which in general form a proper class. However, this only takes place within the axioms that the data are expected to fulfil, but does not enter the way in which the latter are given beforehand, which is exclusively based on the basic opens: that is, the elements of the set A. Any such quantification, moreover, is unproblematic as long as it is not used, for example, to define sets by separation without any further restriction.

If $\lhd$ happens to be a Stone covering, as it is the case for the formal Zariski topology (and presumably for all coverings occurring in a sufficiently concrete algebraic context), then the quantification discussed above can anyway be restricted to the finite subsets of A, which do form a set because A is a set. Likewise, although the products of the set-indexed families of sets that occur in Definition 2 are again sets, in the case of the formal Zariski topology or of an arbitrary Stone covering $\lhd$ it suffices to consider products over finite families of this kind.

As yet, we indeed have only dealt with the sections of a (pre)sheaf $\mathcal{F}$ over the basic opens $a \in A$, together with the corresponding restriction mappings. We stress that only these data have entered the definition of $\mathcal{F}$ being a (pre)sheaf on a formal topology, as it is the case with a sheaf on a basis of opens sets for an ordinary topological space. According to the familiar method to extend a sheaf on a basis to one on the whole topology (see, for instance, [24, p. 69]), we nevertheless define the sections of a (pre)sheaf $\mathcal{F}$ over an arbitrary open $U \subset A$, and also the adequate restriction mappings.

To this end, let $\mathcal{F}$ first be a presheaf, and define $\mathcal{F}(U)$ for $U \subset A$ so that

$$\mathcal{F}(U) \underset{I_U}{\hookrightarrow} \prod_{b \in U} \mathcal{F}(b) \underset{Q_U}{\overset{P_U}{\rightrightarrows}} \prod_{(c,d) \in U \times U} \mathcal{F}(cd)$$

is an equaliser diagram with P_U and Q_U satisfying

$$P_U\left((t_b)_{b\in U}\right)_{(c,d)} = t_c|_{cd}, \qquad Q_U\left((t_b)_{b\in U}\right)_{(c,d)} = t_d|_{cd}$$

for $t_b \in \mathcal{F}(b)$ with $b \in U$ and $(c, d) \in U \times U$. In other words, we define

$$\mathcal{F}(U) = \left\{ (t_b)_{b\in U} \in \prod_{b\in U} \mathcal{F}(b) : P_U\left((t_b)_{b\in U}\right) = Q_U\left((t_b)_{b\in U}\right) \right\}$$

as the kernel of the double arrow consisting of P_U and Q_U, and let I_U denote the corresponding inclusion mapping. If $U = \{a\}$ is a singleton, then $P_U = Q_U$; whence $\mathcal{F}(\{a\}) = \mathcal{F}(a)$ as one surely expects.

Assume next that $\mathcal{F}$ is a sheaf, and let U, V be subsets of A with $V \lhd U$. To define a mapping from $\mathcal{F}(U)$ to $\mathcal{F}(V)$, observe that the composition

$$\mathcal{F}(U) \underset{I_U}{\to} \prod_{b\in U} \mathcal{F}(b) \underset{(r_{ab,b})_{b\in U}}{\to} \prod_{b\in U} \mathcal{F}(ab)$$

[3] In particular, the single arrow i_a is a monomorphism.

factors through i_a for each $a \in V$, because $\mathcal{F}$ is a sheaf and $a \lhd U$. Hence it induces a mapping

$$\mathcal{F}(U) \to \prod_{a \in V} \mathcal{F}(a),$$

which in turn factors through I_V, by the definition of the latter, and thus yields a mapping from $\mathcal{F}(U)$ to $\mathcal{F}(V)$ as required. We thus have achieved the family of mappings

$$R_{V,U} : \mathcal{F}(U) \to \mathcal{F}(V) \quad (V \lhd U),$$

for which $R_{\{a\},\{b\}} = r_{a,b}$ whenever $a \lhd b$, and $R_{V,U}$ stems from the canonical projection whenever $V \subset U$. Moreover, the sheaf properties of $\mathcal{F}$ carry over to the context of arbitrary opens: we have

$$R_{U,U} = \mathrm{id}_{\mathcal{F}(U)} \quad \text{and} \quad U \lhd V \lhd W \Longrightarrow R_{U,W} = R_{U,V} \circ R_{V,W}$$

for all $U, V, W \subset A$, and

$$\mathcal{F}(V) \to \prod_{b \in U} \mathcal{F}(Vb) \rightrightarrows \prod_{c,d \in U} \mathcal{F}(Vcd) \tag{2}$$

is an equaliser diagram for all $U, V \subset A$ with $V \lhd U$, where the arrows once more originate in the appropriate restriction mappings. In particular, we have

$$U \cong V \implies \mathcal{F}(U) \cong \mathcal{F}(V),$$

where $U \cong V$ denotes $U \lhd V \wedge V \lhd U$ as in [41], and $\mathcal{F}(U) \cong \mathcal{F}(V)$ stands for the existence of a group isomorphism. Moreover, we again may use $s|_V$ in place of $R_{V,U}(s)$ whenever $s \in \mathcal{F}(U)$ and $V \lhd U$; and (1) is the special case $V = \{a\}$ of (2). Note, finally, that the families $\mathcal{F}(U)$ and $R_{V,U}$ are indexed by a class.

In all, we have demonstrated how a sheaf on a formal topology gives rise to a sheaf on the corresponding frame of opens, and that if the latter is restricted to the basic opens, then the former is regained. Conversely, if one restricts a sheaf on this frame to the basic opens, then one achieves a sheaf on the given formal topology, and if one extends the latter to the arbitrary opens, then one regains the former up to isomorphism. (This is readily seen from the sheaf condition characteristic of the former, and therefore left to the reader.)

In parallel to [22, V.1.7] all this may serve as a justification our choice of the definition of a sheaf on a formal topology (Definition 2). Why, however, did we not use the name "presheaf on *a basis* of a formal topology" for the object defined in Definition 2? The reason is that, although the monoid underlying a formal topology is a basis of the corresponding frame of opens, the formal topology is the primitive concept, and thus has priority over the frame of opens derived from it.

Let us underline once more that both the initial data and the defining properties of a (pre)sheaf on a formal topology concern only sections over the *basic* opens. There is no need to speak of sections over *arbitrary* opens until one wishes to extend a presheaf to the arbitrary opens, which extension requires that it already is a sheaf in our sense that again involves basic opens only.

It is further noteworthy that we have only made use of restriction mappings of a particular kind, namely, from $\mathcal{F}(a)$ to $\mathcal{F}(ab)$ for $a, b \in A$. This observation helps when one wants to index the restriction mappings without reference to the covering, which works because A is a monoid. Indeed, one could instead provide the restriction mappings as a family

$$\rho_b(a) : \mathcal{F}(a) \to \mathcal{F}(ab) \quad (a, b \in A)$$

with

$$\rho_1 = \mathrm{id} \quad \text{and} \quad \rho_{bc} = \rho_c \circ \rho_b$$

for all $a, b, c \in A$, where the composition $\circ$ is to be understood componentwise with the appropriate index shift:

$$(\rho_c \circ \rho_b)(a) : \mathcal{F}(a) \xrightarrow{\rho_b(a)} \mathcal{F}(ab) \xrightarrow{\rho_c(ab)} \mathcal{F}(abc).$$

(Note that $\rho_b \circ \rho_c = \rho_{cb} = \rho_{bc} = \rho_c \circ \rho_b$, because A is a commutative monoid.) The whole approach to (pre)sheaves could then be carried out as above; in particular, one would have the induced mappings

$$\mathcal{F}(b) = \mathcal{F}(\{b\}) \to \mathcal{F}(\{a\}) = \mathcal{F}(a) \quad (a \lhd b)$$

and thus regain restriction mappings $r_{a,b}$ as before. For the sake of simplicity we have not followed this road, although the $\rho_b(a)$ can easily be defined in the crucial case of the structure sheaf on the formal Zariski topology of a commutative ring, which we will approach soon.

For the time being, the notion of positivity does not occur in our concept of a (pre)sheaf. From a positivity predicate pos, one could require that pos (a) holds whenever $\mathcal{F}(a)$ is a nontrivial group. This would positively express

$$a \lhd \emptyset \implies \mathcal{F}(a) \text{ is trivial}, \tag{3}$$

which follows from the fact that (1) for $U = \emptyset$ is an equaliser diagram: the nullary product in the category of abelian groups is the trivial group. As yet we have no clue whether it is really necessary to ask for the contrapositive of (3), put in a positive way, let alone how to generalise any such request to a binary positivity $\ltimes$.

Definition 3. A *(pre)sheaf of rings* on a formal topology is a (pre)sheaf $\mathcal{F}$ such that $\mathcal{F}(a)$ is a commutative ring for every basic open a, and all restriction mappings are ring homomorphisms. Given (pre)sheaves of rings $\mathcal{F}$ and $\mathcal{G}$ on the same formal topology, a *morphism* $\varphi : \mathcal{F} \to \mathcal{G}$ *of (pre)sheaves of rings* is a family, indexed by the basic opens a, of ring homomorphisms $\varphi(a) : \mathcal{F}(a) \to \mathcal{G}(a)$ that are compatible with the restriction mappings.

The condition involving unary positivity one could furthermore impose means in the particular case of a (pre)sheaf of rings that

$$\mathcal{F}(a) \text{ is nontrivial} \implies \text{pos}(a) \tag{4}$$

for all basic opens a, which positively expresses that the sections over the empty set form a trivial ring.

Let A be a commutative ring, and look first at the structure sheaf on Spec (A) in the classical setting. Its ring of sections over each basic open set $D(a)$ is isomorphic to the ring of fractions

$$A_a = \left\{ \frac{x}{a^n} : x \in A,\ n \in \mathbb{N}_0 \right\}$$

with denominators in the multiplicative subset $M(a)$ generated by a. Nothing appears to hinder us from taking this description as a definition on the formal side, following [22, V.3.3]. We thus also circumvent the – somewhat involved – classical route, for which we refer to [19, II.2].

To achieve the *structure sheaf* $\mathcal{O}_A$ also on the formal Zariski topology of A, let us therefore set the rings of sections as

$$\mathcal{O}_A(a) = A_a \quad (a \in A),$$

for which we clearly have

$$\mathcal{O}_A(1) = A_1 \cong A.$$

We further define the restriction mappings

$$r_{a,b} : \mathcal{O}_A(b) \to \mathcal{O}_A(a) \quad (a \lhd b)$$

as in Corollary 2 (recall that $a \lhd b$ means $a \in R(b)$).

It is easy to see that $\mathcal{O}_A$ is a presheaf of rings, because of the canonical character of the $r_{a,b}$. How can one prove that $\mathcal{O}_A$ is a sheaf? Since the formal Zariski topology comes with a Stone covering, it suffices to prove that (1) is an equaliser diagram for every finite subset U of A. This, however, amounts to verifying that diagram (9) from [24, pp. 125–6] is an equaliser, whose proof is constructive and predicative.

Condition (3) is trivially satisfied for $\mathcal{O}_A$ in place of $\mathcal{F}$, which in this particular case can even be seen without invoking nullary products. In fact, $a \lhd \emptyset$ means $a \in R(0)$, which is equivalent to $0 \in M(a)$ and thus to A_a being trivial. If A_a is nontrivial, then $M(a) \subset \sim \{0\}$ or, equivalently, $\text{Pos}_0(a)$; whence condition (4) is satisfied for the positivity predicate Pos_0 considered in [41]. If A comes with the denial inequality, then (4) for Pos_0 follows from (3), of which it is the contrapositive.

Condition (4) is thus satisfied for Pos whenever this predicate coincides with Pos_0, which – if $\neq$ is the denial inequality on A – is the case precisely when Pos_0 is monotone [41]. In general, and even for $\neq$ as the denial inequality, we cannot see how to constructively deduce $\text{Pos}(a)$ from A_a being nontrivial, let alone from $M(a) \subset \sim \{0\}$: how would the latter fact help to find a power coideal containing a? Working classically, one could simply take $\neg\sqrt{0}$ as a witness of this kind.

4. Sheaves of local rings

Although the stalks of the structure sheaf of the formal Zariski topology are local rings, one cannot accept this property as a definition in the point-free setting of formal topology. We are nonetheless fortunate inasmuch as the well-known point-free concept of a sheaf of local rings carries over to the formal setting.

Since $\lhd$ is reflexive and transitive, the monoid A of a formal topology, with $\cong$ as equality, is partially ordered by $b \leq a$ whenever $a \lhd b$. With respect to this partial order, every formal point ξ is a directed set. So if $\mathcal{F}$ is a presheaf on A, then the restriction mappings

$$r_{a,b} : \mathcal{F}(b) \to \mathcal{F}(a) \quad (a, b \in \xi,\ a \lhd b)$$

form a directed family of arrows; whence their direct limit

$$\mathcal{F}_\xi = \varinjlim_{a \in \xi} \mathcal{F}(a)$$

exists, and is again an abelian group. Following a time-honoured tradition, we call $\mathcal{F}_\xi$ the *stalk*, and its elements the *germs* (of the sections) of $\mathcal{F}$ at ξ.

We now return to the formal Zariski topology of a commutative ring A. Since the formal points are precisely the prime filters of A [41], every formal point ξ gives rise to the ring of fractions $\xi^{-1}A$ that is nontrivial with respect to the denial inequality, and local. Up to isomorphism, this ring is the stalk $\mathcal{O}_{A,\xi}$ of the structure sheaf $\mathcal{O}_A$ at ξ: there is a canonical bijection

$$\mathcal{O}_{A,\xi} \xrightarrow{\cong} \xi^{-1}A;$$

in particular, $\mathcal{O}_{A,\xi}$ is a nontrivial local ring. (This fact, which will be of no relevance below, readily follows from the universal properties of direct limits and rings of fractions.)

In the case of the classical Zariski spectrum Spec (A) [19, II.2.2a], the stalk of the structure sheaf at each point $\mathfrak{p}$ is isomorphic to the ring $A_\mathfrak{p} = (A \setminus \mathfrak{p})^{-1}A$ with the only maximal ideal $\mathfrak{p}A_\mathfrak{p}$—that is, the ideal of $A_\mathfrak{p}$ that is generated by all the fractions $\frac{x}{1}$ with $x \in \mathfrak{p}$. As one then says that the structure sheaf is a sheaf of local rings, it is natural to ask whether one can carry this over to the formal setting.

The traditional definition of a local ring – as possessing a unique maximal ideal – is indeed classically equivalent to the ring being nontrivial, and local. Constructively, on the other hand, the latter concept of a local ring is contained in the former: in a nontrivial local ring in our sense, in which the invertible elements form a prime filter, the noninvertible elements form an ideal, which is the largest proper ideal and thus the only maximal one.

One could now legitimately state that $\mathcal{O}_A$ is a sheaf of (nontrivial) local rings, and this assertion would not cause any problem at all—as long as one did not worry about the existence of formal points. From every point-free perspective, however, a sheaf of local rings on a formal topology can hardly be defined as one whose stalks are all local rings: one can neither be sure that there is at least one formal point nor that the formal points form a set. The case is as for prime ideals (see Section 1), with which prime filters, and thus formal points, are in a one-to-one correspondence.

We wish to stress that any such reservation arises regardless of the fact that the notion of a local ring can well be defined without invoking "the unique maximal ideal". However, there also is the well-known sheaf version of this modified definition, which constructively strengthens its traditional counterpart. As a literal translation of the notion of a local ring in a topos [20, 6.51], it allows us to speak of a sheaf of local rings also in the context of formal topology.

Definition 4. (a) A *sheaf of local rings* on a formal topology is a sheaf of rings $\mathcal{F}$ such that for every basic open a and all $s, t \in \mathcal{F}(a)$ with $s + t \in \mathcal{F}(a)^*$ there is an arbitrary open U with $a \lhd U$ such that

$$s|_{ab} \in \mathcal{F}(ab)^* \vee t|_{ab} \in \mathcal{F}(ab)^* \tag{5}$$

for every $b \in U$.

(b) A *sheaf of nontrivial rings* on a formal topology is a sheaf of rings $\mathcal{F}$ such that

$$\mathcal{F}(a) \text{ is trivial} \implies a \lhd \emptyset \tag{6}$$

for all basic opens a.

Condition (5) is trivially satisfied, by (3), for all the $b \in U$ with $ab \lhd \emptyset$, but also, with $U = \emptyset$, if already $a \lhd \emptyset$. Note also that (6) is the converse of (3). Concerning the occurrence of a quantification over arbitrary opens in the definition of a sheaf of local rings we refer to the discussion following Definition 2.

In the presence of a positivity predicate pos, a sheaf of nontrivial rings $\mathcal{F}$ is further required to satisfy

$$\mathrm{pos}(a) \implies \mathcal{F}(a) \text{ is nontrivial.} \tag{7}$$

Condition (7) reflects the idea that the ring of sections over an inhabited basic open set be nontrivial, and it implies (6) whenever pos satisfies "*ex falso quodlibet*". As yet we have no clue how to generalise (7) to the case of a binary positivity $\ltimes$; the case is as with its converse (4).

Let now A be a commutative ring. The structure sheaf $\mathcal{O}_A$ is a sheaf of nontrivial local rings, in the sense of Definition 4, on the formal Zariski topology of A. In fact, condition (6) with $\mathcal{O}_A$ in place of $\mathcal{F}$ holds as trivially as its converse (3) does for the same data; whence $\mathcal{O}_A$ is a sheaf of nontrivial rings. To see that $\mathcal{O}_A$ is a sheaf of local rings, let $a \in A$ and $s, t \in A_a$ with $s = \frac{b}{a^n}$ and $t = \frac{c}{a^m}$. We show that if $s + t \in (A_a)^*$, then $U = \{b, c\}$ is as required. First, $s + t = \frac{ba^m + ca^n}{a^{m+n}}$ is invertible in A_a precisely when there is $d \in A$ such that $d(ba^m + ca^n) \in M(a)$, which is to say that $d(ba^m + ca^n) = a^k$ for some $k \in \mathbb{N}_0$. Hence if $s + t \in (A_a)^*$, then $a \in R(b, c)$ or, equivalently, $a \lhd U$. Secondly, $s|_{ab} = \frac{b^{n+1}}{(ab)^n}$ and $t|_{ac} = \frac{c^{m+1}}{(ac)^m}$ are invertible in A_{ab} and A_{ac}, respectively, because $a^{n+1}b^{n+1} \in M(ab)$ and $a^{m+1}c^{m+1} \in M(ac)$.

Is $\mathcal{O}_A$ also a sheaf of nontrivial rings with respect to positivity? If pos is weakly monotone, then condition (7) with $\mathcal{O}_A$ in place of $\mathcal{F}$ follows from (6) for the same data provided that A comes with the denial inequality. In particular, (7) holds for Pos_0 – and thus for Pos – provided that $\neq$ is the denial inequality.

Unlike (4), which is hard to fulfill in general even when $\neq$ is the denial inequality, condition (7) will be invoked in the following. However, this will only happen in the presence of positivity, which we do not need to suppose anyway. So one could now do without (7), and leave the notion of an inequality as unspecified. As from this point an inequality independent of equality will not be essential either, we may assume that *every commutative ring which will occur in the sequel comes with $\neq$ as the denial inequality.*

5. Local morphisms without points

By now, we have established in a point-free way the structure sheaf $\mathcal{O}_A$ on the formal Zariski topology as a sheaf of nontrivial local rings which, according to its very construction, solves the representation problem of representing a commutative ring A as the ring of global sections $\mathcal{O}_A(1)$. We next turn our attention to the universal property that makes the Zariski spectrum stand out among the locally ringed spaces. A standard reference for the classical approach, which we recall first, is [19, II.2].

Classically, a (locally) ringed space $(X, \mathcal{O}_X)$ is a topological space X with a sheaf $\mathcal{O}_X$ of (local) rings, whose sections usually stand for a certain type of – possibly generalised – functions on open subsets of X. So a ringed space $(X, \mathcal{O}_X)$ is a locally ringed space if and only if the stalk $\mathcal{O}_{X,x}$ of the sheaf $\mathcal{O}_X$ at each point $x \in X$ is a local ring with the one and only maximal ideal $\mathfrak{m}_x = \mathcal{O}_{X,x} \setminus \mathcal{O}^*_{X,x}$.

A morphism of ringed spaces $(f, \varphi) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ consists of a continuous mapping $f : X \to Y$ and a morphism $\varphi : \mathcal{O}_Y \to \mathcal{O}_X \circ f^{-1}$ of sheaves of rings on Y, which is—possibly an analogue of—the lifting of functions along f. If both $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are *locally* ringed spaces, then (f, φ) is even a morphism of *locally* ringed spaces whenever, in addition, the ring homomorphism $\varphi_x : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ induced by φ is local for all $x \in X$ and $y = f(x)$. This means that $\varphi_x^{-1}(\mathfrak{m}_x) \supset \mathfrak{m}_y$ holds for the respective maximal ideals—or, equivalently, $\varphi_x^{-1}(\mathfrak{m}_x) = \mathfrak{m}_y$.

The universal property of Spec (A) – together with the traditional Zariski topology and the associated structure sheaf $\mathcal{O}_{\mathrm{Spec}(A)}$ – consists in the presence of the canonical isomorphism

$$\mathrm{Hom}\,(A, \mathcal{O}_X(X)) \cong \mathrm{Hom}\left((X, \mathcal{O}_X), (\mathrm{Spec}\,(A)\,, \mathcal{O}_{\mathrm{Spec}(A)})\right),$$

which is functorial in the variable locally ringed space $(X, \mathcal{O}_X)$. In other words, the locally ringed space $(\mathrm{Spec}\,(A)\,, \mathcal{O}_{\mathrm{Spec}(A)})$ represents the set-valued contravariant functor on the category of locally ringed spaces that assigns the set $\mathrm{Hom}\,(A, \mathcal{O}_X(X))$ to each $(X, \mathcal{O}_X)$. To define this isomorphism from right to left, one simply picks the global sections of the “lifting of functions”; the reverse direction requires some dealing with points and ideals—even in the otherwise point-free localic approach [22, V.3.5].

Can we nonetheless speak of that universal property in point-free terms—and, if so, how? As we have already seen, the notion of a local ring can be expressed in a straightforward way without any talk of “the unique maximal ideal”; it is well-known that one can equally do so with that of a local ring homomorphism: a ring homomorphism $\psi : A \to B$ is called *local* whenever it is strongly extensional with respect to the natural inequality: that is, $\psi^{-1}(B^*) \subset A^*$ or, equivalently, $\psi^{-1}(B^*) = A^*$. This definition of a local ring homomorphism generalises the traditional one inasmuch as none of the rings under consideration is supposed to be local from the outset, and classically collapses to the latter whenever both A and B are local. How can the alternative concept be extended to the context of sheaves?

The point-free concept of a sheaf of local rings, which we have remembered before, and thus tacitly that of a locally ringed space are relatively common. The renewed definition of a local ring homomorphism gave rise to a topos-theoretic version [20, 6.51]. In the sequel, we will introduce the notion of a morphism of locally ringed spaces to formal topology. Once more, the clue is to focus on the positive concept of invertibility rather than its negation. Since only the former is a truly local property,[4] any such way to grasp locality is also closer to the bare meaning of this word than the traditional approach based on points.

To start with, observe first that, for each ringed space $(X, \mathcal{O}_X)$, the family of supports

$$\vartheta_U : \mathcal{O}_X(U) \to \Omega(U)\,, \; s \mapsto \{x \in U : s_x \in \mathcal{O}^*_{X,x}\} \quad (U \subset X \text{ open}) \tag{8}$$

naturally links the algebraic part with the topological part of this ringed space, where $\Omega(U)$ stands for the lattice of open subsets of U and s_x for the germ at a point x of U of a section s over U. Moreover, $(X, \mathcal{O}_X)$ is a *locally* ringed space if and only if ϑ_U is a support in the lattice-theoretic sense (see below) for every open $U \subset X$. This is the case if, for instance, $(X, \mathcal{O}_X)$ equals $\left(\mathrm{Spec}\,(A)\,, \mathcal{O}_{\mathrm{Spec}(A)}\right)$ for a commutative ring A, in which case $\vartheta_X(a) = D(a)$ for every $a \in A$ when one identifies A with the ring of global sections $\mathcal{O}_X(X)$ isomorphic to it.

The distinctive property of a morphism $(f, \varphi) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of *locally* ringed spaces amounts to

$$\vartheta_U(\varphi(s)|_U) \subset f^{-1}(\vartheta_V(s)) \tag{9}$$

for all open $V \subset Y$, $s \in \mathcal{O}_Y(V)$, and open $U \subset X$ with $U \subset f^{-1}(V)$. Indeed, (9) is equivalent to

$$\varphi(s)_x \in \mathcal{O}^*_{X,x} \Longrightarrow s_y \in \mathcal{O}^*_{Y,y} \tag{10}$$

for all $x \in U$ with $y = f(x)$. As the converse of (10) holds trivially, (9) can equivalently be expressed as

$$\vartheta_U(\varphi(s)|_U) = U \cap f^{-1}(\vartheta_V(s))\,. \tag{11}$$

The formal counterpart of this essentially point-free condition will serve us later to redefine the locality of morphisms without invoking points.

4 This becomes particularly visible when one looks at the standard example of a locally ringed space: that of a topological space X with $\mathcal{O}_X$ as the sheaf of continuous real-valued functions on it, for which $\mathfrak{m}_x = \{s \in \mathcal{O}_{X,x} : s(x) = 0\}$ for every $x \in X$. In this context, φ_x is nothing but the lifting of germs of functions along f; whence $\varphi_x^{-1}(\mathfrak{m}_x) = \mathfrak{m}_y$ amounts to the trivial condition that $s \circ f(x) = 0$ if and only if $s(y) = 0$. Moreover, $s \in \mathcal{O}^*_{X,x}$ means nothing but $s(x) \neq 0$, which in fact is a local property of x.

5.1. Morphisms of formal topologies

We recall that a *morphism* $F : A \to B$ *of formal topologies* with underlying monoids A and B is a family

$$F(a) \quad (a \in A)$$

of subsets of B subject to the following conditions [35]. First, F is expected to satisfy

$$1 \lhd F(1) \quad \text{and} \quad F(c)\,F(e) \lhd F(ce) \tag{12}$$

for all $c, e \in A$. Next, F has to preserve the covering relations: that is,

$$a \lhd U \implies F(a) \lhd F(U) \tag{13}$$

for all $a \in A$ and $U \subset A$, with the notation

$$F(U) = \bigcup \{F(c) : c \in U\}. \tag{14}$$

Note that (13) implies

$$U \lhd V \implies F(U) \lhd F(V) \tag{15}$$

for all $U, V \subset A$.

If A and B are endowed with positivity predicates, then F is assumed to reflect them: that is,

$$\mathrm{pos}\,(F(a)) \implies \mathrm{pos}\,(a) \tag{16}$$

for all $a \in A$. Since (13) amounts to

$$a \lhd U \implies F(a) \lhd \{b \in B : \exists c \in A \ (b \in F(c) \wedge c \in U)\},$$

in the presence of positivity relations on A and B condition (16) has to be generalised to

$$F(a) \ltimes \{b \in B : \forall c \in A\,(b \in F(c) \Rightarrow c \in U)\} \implies a \ltimes U. \tag{17}$$

Clearly, (16) is the special case $U = A$ of (17), which is equivalent to the condition given in [17] (see also [16]).

All these conditions perfectly mirror the properties of the inverse image operator $F = f^{-1}$ that is associated with a continuous mapping f. On the other hand, each morphism $F : A \to B$ of formal topologies can be extended, by (14), to a frame homomorphism $\mathrm{Open}\,(A) \to \mathrm{Open}\,(B)$; this is well-defined because

$$U \cong V \implies F(U) \cong F(V) \tag{18}$$

for all $U, V \subset A$ by (15). Moreover, each occurrence of $\lhd$ in (12) can equivalently be replaced by one of $\cong$, by virtue of (15) and the properties of a covering relation, so that (12) is equivalent to the request that F be a monoid homomorphism up to $\cong$: that is,

$$F(1) \cong 1 \quad \text{and} \quad F(ce) \cong F(c)\,F(e) \tag{19}$$

for all $c, e \in A$. In particular, if $\mathcal{B}$ is a sheaf on B, then $\mathcal{B} \circ F$ with

$$\mathcal{B} \circ F(a) = \mathcal{B}(F(a)) \quad (a \in A)$$

is a sheaf on A.

In [34] it was proposed to assume that every morphism F be *saturated*, which is to say that every $F(a)$ is a saturated subset of B. If only for the sake of a simpler presentation, we do not follow this proposal, but then have to consider two morphisms of formal topologies $F, G : A \to B$ as *equal*, for short $F \cong G$, whenever $F(a) \cong G(a)$ for all $a \in A$.

By a *basic monoidal formal topology* we understand a commutative monoid together with a covering relation $\lhd$ and a positivity relation $\ltimes$. An object of the same kind but without any positivity structure at all will simply be called a *monoidal formal topology*.[5] With the appropriate notion of a morphism as above, the (basic) monoidal formal topologies form a category, for which we use the shorthand (**B**)**MFTop**. Note that there is a forgetful functor from **BMFTop** to **MFTop**.

In the following, each commutative ring A is tacitly equipped with the structure of its formal Zariski topology, which object in **BMFTop** – or in **MFTop** – we equally denote by A. We likewise call an arbitrary monoidal formal topology by its underlying monoid.

5 Basic formal topologies are less basic, but the notion of a positivity relation stems from the basic picture [16,35–38].

5.2. Supports

According to [23] (see also [44,8]), a *support* on a commutative ring A with values in a (bounded) lattice L is a mapping $F : A \to L$ such that

$$F(1) = 1,\ F(ab) = F(a) \wedge F(b), \tag{20}$$

$$F(0) = 0,\ F(a+b) \leq F(a) \vee F(b) \tag{21}$$

for all $a, b \in A$. If $F : A \to L$ is a support, then so is $\lambda \circ F \circ \varphi$ whenever $\varphi : A' \to A$ and $\lambda : L \to L'$ are homomorphisms of rings and lattices, respectively.

As the frame of arbitrary opens $\mathrm{Open}(B)$ of a formal topology B consists of all subsets of the underlying monoid, in general it is a proper class. In the following we understand every family $F(a)$ of subsets of B with $a \in A$ as a mapping $F : A \to \mathrm{Open}(B)$, and can thus formulate the following characterisation.

Lemma 3. *Let A be a commutative ring and $B \in$ **MFTop**. A morphism of formal topologies $F : A \to B$ is nothing but a support $F : A \to \mathrm{Open}(B)$, and conversely.*

Proof. Observe first that for the multiplicative properties (19) and (20) there is nothing to prove. Let next $F : A \to B$ be a morphism of formal topologies. As $0 \lhd \emptyset$, we have $F(0) \lhd F(\emptyset)$, and thus $F(0) \cong \emptyset$ because $F(\emptyset) = \emptyset$ and $\emptyset \lhd F(0)$ anyway. If $a, c \in A$, then $a + c \lhd \{a, c\}$, and thus $F(a+c) \lhd F(a) \cup F(c)$ as required.

As for the converse, let $F : A \to \mathrm{Open}(B)$ be a support. If $a \lhd U$ for $a \in A$ and $U \subset A$, then $a^n = \sum_{i=1}^m r_i b_i$ with $n \in \mathbb{N}$, $m \in \mathbb{N}_0, b_1, \ldots, b_m \in U$, and $r_1, \ldots, r_m \in A$. Hence $F(a^n) \lhd \bigcup_{i=1}^m F(r_i b_i)$, and thus $F(a) \lhd F(U)$: indeed, $F(a) \cong F(a^n)$, and $F(r_i b_i) \cong F(r_i) F(b_i) \lhd F(b_i)$. $\square$

Here is an alternative interpretation of Lemma 3. Given a commutative ring A, the Zariski frame of the radical ideals of A is equal to $\mathrm{Sat}(A)$ and thus isomorphic to $\mathrm{Open}(A)$. More specifically, the inclusion mapping $\mathrm{Sat}(A) \subset \mathrm{Open}(A)$ is an isomorphism whose inverse is the radical operator $R : \mathrm{Open}(A) \to \mathrm{Sat}(A)$ that assigns the radical $R(U)$ of the ideal $I(U)$ generated by U to each $U \in \mathrm{Open}(A)$; this is because elements U, V of $\mathrm{Open}(A)$ are considered to be equal, for short $U \cong V$, precisely when $R(U) = R(V)$.

Again in [8], the *radical support* on A with values in its Zariski frame is defined as the mapping $R : A \to \mathrm{Sat}(A)$ that assigns the radical $R(a)$ of the principal ideal $I(a)$ to each $a \in A$. It has the universal property that if L is a frame and $F : A \to L$ is a support, then there is a unique frame homomorphism $\lambda : \mathrm{Sat}(A) \to L$ with $\lambda \circ R = F$.

In particular, if A is a commutative ring and B is a formal topology, then each support $F : A \to \mathrm{Open}(B)$ uniquely determines a frame homomorphism $\mathrm{Open}(A) \to \mathrm{Open}(B)$ that maps $\{a\} \cong R(a)$ to $F(a)$ for every $a \in A$. As a consequence of Lemma 3, this homomorphism between the frames of opens stems from a morphism of the underlying formal topologies—namely, the one given by the support F.

Before we proceed, we carry Lemma 3 over to the context of positivity. By a *frame with predicate* we understand a frame L with a distinguished predicate pos. The frame $\mathrm{Open}(B)$ together with pos, for a formal topology B with a positivity predicate pos, will be the only example of a frame with predicate occurring in the sequel. We refer to [30] for a more detailed treatment, where a frame with predicate is called a "frame with apartness" whenever this predicate has the properties that are analogous to monotonicity and openness of a positivity predicate.

When L is a frame with predicate, we expect every support $F : A \to L$ to satisfy the additional condition

$$\mathrm{pos}(F(a)) \implies a \neq 0 \tag{22}$$

for all $a \in A$. Since $F(a^n) = F(a)$ for all $n \in \mathbb{N}$, this form of strong extensionality for supports is equivalent to

$$\mathrm{pos}(F(a)) \implies \mathrm{Pos}_0(a) \tag{23}$$

for all $a \in A$. In particular, Lemma 3 remains valid with the positivity predicate Pos_0 from [41] on A and an arbitrary positivity predicate pos on B.

Moreover, Lemma 3 even holds in the presence of a positivity relation on B. To see this, let A be a commutative ring and $B \in$ **BMFTop**. We show that the morphisms $F : A \to B$ in **BMFTop** coincide with the supports $F : A \to \mathrm{Open}(B)$ that satisfy (22) with Pos in place of pos. On the one hand, if $F : A \to B$ is a morphism in **BMFTop**, and $a \in A$, then $\mathrm{Pos}(F(a))$ implies $\mathrm{Pos}(a)$ by (16); since $\mathrm{Pos}(a)$ means that a is contained in a power coideal of A, this implies $a \neq 0$ as required.

On the other hand, let $F : A \to \mathrm{Open}(B)$ be a support satisfying (22) with Pos in place of pos, and assume that the antecedent of (17) is valid: that is, $F(a) \ltimes W$ or, equivalently, $a \in Q$ with

$$W = \{b \in B : \forall c \in A\,(b \in F(c) \Rightarrow c \in U)\} \quad \text{and} \quad Q = \{c \in A : F(c) \ltimes W\}.$$

If $F(c) \ltimes W$, then $F(c) \between W$, and thus $c \in U$; in particular, $Q \subset U$. To establish Q as a witness for $a \ltimes U$, it thus suffices to show that Q is a power coideal.

To start with, if $F(c) \ltimes W$, then $\mathrm{Pos}(F(c))$; whence $c \neq 0$ by hypothesis. By the compatibility of $\ltimes$ with $\lhd$ we achieve the other properties of a power coideal. First, if $F(c+e) \ltimes W$, then either $F(c) \ltimes W$ or $F(e) \ltimes W$ because $F(c+e) \lhd F(c) \cup F(e)$. Next, if $F(ce) \ltimes W$, then $F(c) \ltimes W$ because $F(ce) \cong F(c)F(e) \lhd F(c)$; likewise $F(e) \ltimes W$. Finally, if $F(c) \ltimes W$, then $F(c^n) \ltimes W$ because $F(c) \cong F(c^n)$.

5.3. The category of formal geometries

If $\lhd$ is a covering relation on a monoid B, and $b \in B$, then $b^{\lhd} = \{d \in B : d \lhd b\}$ carries the induced covering relation, which suffices to define $\mathrm{Open}(b^{\lhd})$; furthermore, $b^{\lhd}$ is endowed with the induced positivity structure whenever there is one on B. Recall also that $1^{\lhd} = B$.

Definition 5. A *(locally) ringed formal topology* $(B, \mathcal{B})$ is a formal topology B together with a sheaf of (nontrivial local) rings $\mathcal{B}$ on B.

For each ringed formal topology $(B, \mathcal{B})$, the family of mappings

$$\Theta_b : \mathcal{B}(b) \to \mathrm{Open}(b^{\lhd}),\ s \mapsto \left\{d \in b^{\lhd} : s|_d \in \mathcal{B}(d)^*\right\} \quad (b \in B) \tag{24}$$

is a perfect point-free substitute for the family of supports ϑ from (8). Indeed, Θ will prove to be as crucial as its forerunner for understanding the "locality" of morphisms. (As every $\Theta_b(s)$ is saturated, one could equally have defined Θ_b as a mapping from $\mathcal{B}(b)$ to $\mathrm{Sat}(b^{\lhd})$, but would then sometimes face unnecessary notational redundancy.)

Lemma 4. *A ringed formal topology $(B, \mathcal{B})$ is a* locally *ringed formal topology if and only if Θ_b is a support for every $b \in B$.*

Proof. Let $b \in B$. We clearly have $\Theta_b(1) = b^{\lhd}$. For all $s, t \in \mathcal{B}(b)$, moreover,

$$\Theta_b(s)\,\Theta_b(t) \underset{\text{(i)}}{\subset} \Theta_b(st) \underset{\text{(ii)}}{=} \Theta_b(s) \cap \Theta_b(t) \underset{\text{(iii)}}{\lhd} \Theta_b(s)\,\Theta_b(t)\,;$$

whence $\Theta_b(st) \cong \Theta_b(s)\,\Theta_b(t)$. To verify (i), let $d, e \in b^{\lhd}$. If $s|_d \in \mathcal{B}(d)^*$ and $t|_e \in \mathcal{B}(e)^*$, then both $s|_{de} = (s|_d)|_{de}$ and $t|_{de} = (t|_e)|_{de}$ belong to $\mathcal{B}(de)^*$, and so does $(s|_{de})(t|_{de}) = (st)|_{de}$. As for (ii), notice for every $d \in b^{\lhd}$ that $(st)|_d = (s|_d)(t|_d)$ is an element of $\mathcal{B}(d)^*$ if and only if so are both $s|_d$ and $t|_d$. To see (iii), simply observe that $d \lhd d^2$.

Next, it is plain that $\mathcal{B}$ is a sheaf of local rings precisely when

$$\forall d \in \Theta_b(s+t)\ \exists V \subset B\ (d \lhd V \wedge dV \subset \Theta_d(s|_d) \cup \Theta_d(t|_d)) \tag{25}$$

for all $b \in B$ and $s, t \in \mathcal{B}(b)$. We show that, for any choice of $b \in B$ and of $s, t \in \mathcal{B}(b)$, condition (25) is equivalent to

$$\Theta_b(s+t) \lhd \Theta_b(s) \cup \Theta_b(t)\,. \tag{26}$$

To achieve this goal it is useful to observe that

$$d\Theta_b(s) \subset \Theta_d(s|_d) \subset \Theta_b(s) \tag{27}$$

whenever $d \lhd b$, and likewise with t in place of s. (If $e \in \Theta_b(s)$, then $de \in \Theta_d(s|_d)$ for $(s|_d)|_{de} = s|_{de} = (s|_e)|_{de}$; if $e \in \Theta_d(s|_d)$, then $e \in \Theta_b(s)$ for $s|_e = (s|_d)|_e$.)

Assume now that (25) holds. If $d \in \Theta_b(s+t)$, then pick V as in (25), for which

$$d \lhd d^2 \lhd dV \subset \Theta_d(s|_d) \cup \Theta_d(t|_d) \underset{(27)}{\subset} \Theta_b(s) \cup \Theta_b(t)$$

as required in (26). As for the converse, suppose that (26) is valid, and set

$$V = \Theta_b(s) \cup \Theta_b(t)\,. \tag{28}$$

If $d \in \Theta_b(s+t)$, then $d \lhd V$ by (26), and

$$dV = d\Theta_b(s) \cup d\Theta_b(t) \underset{(27)}{\subset} \Theta_d(s|_d) \cup \Theta_d(t|_d)$$

as required in (25).

Finally, if $d \lhd b$, then the ring $\mathcal{B}(d)$ is trivial precisely when $d \in \Theta_b(0)$. With this at hand it is easy to see that

$$\mathcal{B}(b) \text{ is trivial} \implies b \lhd \emptyset$$

for all $b \in B$ if and only if

$$d \in \Theta_b(0) \implies d \lhd \emptyset$$

for all $b, d \in B$. In other words, $\mathcal{B}$ is a sheaf of nontrivial rings precisely when $\Theta_b(0) \lhd \emptyset$, which is to say that $\Theta_b(0) \cong \emptyset$. $\square$

The proof of Lemma 4 shows, for the defining conditions that Θ_b be a support for every $b \in B$, that the multiplicative part (20) is satisfied for any ringed formal topology $(B, \mathcal{B})$ whatsoever, and that only the additive part (21) is crucial for $\mathcal{B}$ being a sheaf of nontrivial local rings on B. This, of course, parallels the classical case. Note also that, with Θ at hand, by (28) the open V required in (25) can be defined uniformly in $d \in \Theta_b\,(s+t)$.

Lemma 4 can further be carried over to the case of a ringed formal topology $(B, \mathcal{B})$ with a positivity predicate pos on B. Indeed, condition (7) is equivalent to

$$\mathrm{pos}\,(d) \implies \left(s|_d \in \mathcal{B}\,(d)^* \Rightarrow s \neq 0\right)$$

for all $b \in B$, $s \in \mathcal{B}\,(b)$, and $d \in b^{\lhd}$, which is to say that $\mathrm{pos}\,(\Theta_b\,(s))$ implies $s \neq 0$ for all $b \in B$ and $s \in \mathcal{B}\,(b)$.

We next conceive the category of formal geometries as a perfect analogue to that of locally ringed spaces, which are sometimes called geometric spaces. [6]

Definition 6. A *formal geometry* is a locally ringed formal topology $(B, \mathcal{B})$ with $B \in$ **MFTop**. Given formal geometries $(A, \mathcal{A})$ and $(B, \mathcal{B})$, a *morphism* $(F, \varphi) : (A, \mathcal{A}) \to (B, \mathcal{B})$ *of formal geometries* consists of a morphism $F : A \to B$ in **MFTop** and a morphism $\varphi : \mathcal{A} \to \mathcal{B} \circ F$ of sheaves of rings on A satisfying the *locality condition*: that is,

$$\Theta_b\,(\varphi\,(a)\,(s)|_b) \lhd F\,(\Theta_a(s)) \tag{29}$$

for all $a \in A$, $s \in \mathcal{A}\,(a)$, and $b \in F\,(a)$.

With the obvious way to compose morphisms of this kind and to set the identity, we thus get the *category* **FGeom** *of formal geometries*. Note that the locality condition (29) is nothing but a practicable way to express in point-free terms the defining condition (9) for a morphism between ringed spaces to be one between *locally* ringed spaces. Of course, the former condition is classically equivalent to the latter provided that the formal topology B has enough points in the sense recalled, for instance, in [15,41].

In a way that is analogous to the passing from (9) to (11), the locality condition (29) can equivalently be put as

$$\Theta_b\,(\varphi\,(a)\,(s)|_b) \cong bF\,(\Theta_a(s))\,. \tag{30}$$

To see this equivalence, observe first that (30) implies (29). Conversely,

$$bF\,(\Theta_a(s)) \subset \Theta_b\,(\varphi\,(a)\,(s)|_b) \lhd bF\,(\Theta_a(s))\,,$$

in which the second step follows from (29), and the first one can be seen as follows. An arbitrary element of $bF\,(\Theta_a(s))$ is of the form bd with $d \in F\,(c)$ for some $c \in \Theta_a(s)$, for which, in particular, $c \lhd a$. Hence $bd \lhd b \lhd F\,(a)$, $bd \lhd d \lhd F\,(c)$, $F\,(c) \lhd F\,(a)$, and $s|_c \in \mathcal{A}\,(c)^*$, so that

$$\varphi\,(a)\,(s)|_{bd} = \varphi\,(c)\,(s|_c)|_{bd} \in \mathcal{B}(bd)^*,$$

which yields $bd \in \Theta_b\,(\varphi\,(a)\,(s)|_b)$ as required.

5.4. *The universal property*

Let **CRings** stand for the *category of commutative rings*. In the rest of this section, we extend the formal Zariski spectrum to a covariant[7] functor

$$\mathrm{FSpec} : \mathbf{CRings} \to \mathbf{FGeom}$$

and show that this is left adjoint to the global section functor.

From now on let A always denote a commutative ring with the denial inequality $\neq$. As the structure sheaf $\mathcal{O}_A$ is a sheaf of nontrivial local rings (Section 4), the *formal Zariski spectrum*

$$\mathrm{FSpec}\,(A) = (A, \mathcal{O}_A)$$

of A is a formal geometry. We thus have defined FSpec on the objects; its definition on the morphisms will be given immediately after, and as a by-product of, the following more general construction.

There is the covariant *global section functor*

$$\Gamma : \mathbf{FGeom} \to \mathbf{CRings}$$

with $\Gamma\,(B, \mathcal{B}) = \mathcal{B}\,(1)$. Given a commutative ring A and a formal geometry $(B, \mathcal{B})$, we construct a mapping

$$\Phi : \mathrm{Hom}_{\mathbf{CRings}}\,(A, \Gamma\,(B, \mathcal{B})) \to \mathrm{Hom}_{\mathbf{FGeom}}\,(\mathrm{FSpec}\,(A)\,, (B, \mathcal{B}))$$

[6] This is for the good reason that most spaces occuring in geometry carry additional structure which gives naturally rise to a sheaf of local rings whose sections are (generalised) functions.

[7] In view of the intended meaning of a morphism between formal topologies as the inverse image operator of a continuous mapping, it is no wonder but worth pointing out that – in contrast to the classical case – the algebraic part of a morphism of formal geometries points to the same direction as its topological part.

by assigning a morphism

$$\Phi(\psi) = (F, \varphi) : \mathrm{FSpec}(A) \to (B, \mathcal{B})$$

of formal geometries to each ring homomorphism $\psi : A \to \mathcal{B}(1)$. To this aim, we first set

$$F(a) = \Theta_1(\psi(a)) \quad (a \in A).$$

As ψ is a ring homomorphism, and Θ_1 a support (Lemma 4), $\Theta_1 \circ \psi : A \to \mathrm{Open}(B)$ again is a support—or, equivalently (Lemma 3), $F : A \to B$ is a morphism of formal topologies.

To continue the construction of Φ, we construct a morphism of sheaves of rings $\varphi : \mathcal{O}_A \to \mathcal{B} \circ F$. We fix $a \in A$ for a while, and recall that $\psi(a)|_b \in \mathcal{B}(b)^*$ for every $b \in F(a)$ by the definition of $F(a)$. Hence, for every $b \in F(a)$,

$$A \underset{\psi}{\to} \mathcal{B}(1) \underset{r_{b,1}}{\to} \mathcal{B}(b)$$

factors through the canonical mapping $A \to A_a$, which is to say that it induces an arrow

$$\mathcal{O}_A(a) = A_a \to \mathcal{B}(b).$$

We thus arrive at a mapping from $\mathcal{O}_A(a)$ to $\prod_{b \in F(a)} \mathcal{B}(b)$, whose range clearly lies in $\mathcal{B}(F(a))$. The resulting family of ring homomorphisms

$$\varphi(a) : \mathcal{O}_A(a) \to \mathcal{B}(F(a)) \quad (a \in A)$$

is by construction compatible with the restriction mappings, and thus defines a morphism of sheaves of rings.

In order that (F, φ) be a morphism of formal geometries, it remains to prove that (F, φ) satisfies the locality condition (29). To this aim, fix $a \in A$, $s \in \mathcal{O}_A(a) = A_a$, and $b \in F(a)$. Let $s = \frac{x}{a^n}$ with $x \in A$ and $n \in \mathbb{N}_0$, and set $c = ax$ in $A = \mathcal{O}_A(1)$; then $s = \frac{c}{a^{n+1}}$ in A_a. We now have $c \lhd a$ and $\frac{a^{n+1}}{c} s|_c = 1$ in $\mathcal{O}_A(c) = A_c$; whence $s|_c \in \mathcal{O}_A(c)^*$ and thus $c \in \Theta_a(s)$. From $c|_a = \frac{a^{n+1}}{1} s$ in A_a we get

$$\psi(c)|_{F(a)} = \varphi(1)(c)|_{F(a)} = \varphi(a)(c|_a) = \varphi(a)\left(\frac{a^{n+1}}{s}\right)\varphi(a)(s)$$

in $\mathcal{B}(F(a))$. Since $\frac{a^{n+1}}{1} \in \mathcal{O}_A(a)^*$, we have $\varphi(a)\left(\frac{a^{n+1}}{1}\right) \in \mathcal{B}(F(a))^*$. For each $d \in \Theta_b(\varphi(a)(s)|_b)$ we also have $\varphi(a)(s)|_d \in \mathcal{B}(d)^*$ and thus arrive at

$$\psi(c)|_d = \left(\psi(c)|_{F(a)}\right)|_d \in \mathcal{B}(d)^*,$$

which amounts to

$$d \in \Theta_1(\psi(c)) = F(c) \subset F(\Theta_a(s)).$$

This shows that (29) is even valid with $\subset$ in place of $\lhd$.

The construction of Φ is now completed, for the given commutative ring A and an arbitrary formal geometry $(B, \mathcal{B})$. If, in particular, also $(B, \mathcal{B}) = \mathrm{FSpec}(B)$ is the formal Zariski spectrum of a commutative ring B, then Φ induces a mapping

$$\mathrm{Hom}_{\mathbf{CRings}}(A, B) \to \mathrm{Hom}_{\mathbf{FGeom}}(\mathrm{FSpec}(A), \mathrm{FSpec}(B)), \ \psi \mapsto \mathrm{FSpec}(\psi),$$

by which we define the functor FSpec on the morphisms: if $\psi \in \mathrm{Hom}_{\mathbf{CRings}}(A, B)$, then

$$\mathrm{FSpec}(\psi)(a) = R(\psi(a))$$

for every $a \in A$ (Lemma 1). It is routine to verify that FSpec is indeed a covariant functor.

Now let $(B, \mathcal{B})$ again stand for an arbitrary formal geometry, and still A for a commutative ring. Recall that we have the canonical ring homomorphism

$$\pi : A \to \Gamma \circ \mathrm{FSpec}(A) = A_1, \ a \mapsto \frac{a}{1}$$

which in fact is an isomorphism (Section 2).

Proposition 5. *Φ is bijective, with its inverse being induced by Γ.*

Proof. Clearly, if $\psi : A \to \mathcal{B}(1)$ is a ring homomorphism and $(F, \varphi) = \Phi(\psi)$, then ψ coincides, by the construction of φ, with the mapping

$$A \underset{\pi}{\overset{\cong}{\to}} A_1 = \mathcal{O}_A(1) \underset{\varphi(1)}{\to} \mathcal{B}(F(1)) \underset{r_{1,F(1)}}{\overset{\cong}{\to}} \mathcal{B}(1) .$$

Hence it suffices to show, for each morphism of formal geometries (G, η) from FSpec (A) to $(B, \mathcal{B})$, that $(G, \eta) = \Phi(\psi)$ where

$$\psi : A \underset{\pi}{\overset{\cong}{\to}} A_1 = \mathcal{O}_A(1) \underset{\eta(1)}{\to} \mathcal{B}(G(1)) \underset{r_{1,G(1)}}{\overset{\cong}{\to}} \mathcal{B}(1) .$$

To this end, let $\Phi(\psi) = (F, \varphi)$, for which $\varphi = \eta$ by the construction of φ. To see that F and G are equal morphisms of formal topologies, fix $a \in A$. By the definitions of F and ψ, we have

$$F(a) = \Theta_1(\psi(a)) = \Theta_1(\eta(1) \circ \pi(a)|_1) .$$

The locality condition (30) with (G, η) in place of (F, φ), which says that

$$\Theta_1(\eta(1) \circ \pi(a)|_1) \cong G(\Theta_1(\pi(a))) ,$$

now yields $F(a) \cong G(\Theta_1(\pi(a)))$. As furthermore $\Theta_1(\pi(a)) = R(a)$ by Lemma 1, and $G(R(a)) \cong G(a)$ because $R(a) \cong a$, we arrive at $F(a) \cong G(a)$. □

Just as its inverse, Φ is functorial in $(B, \mathcal{B})$ and in A. This observation, whose routine verification we may omit, allows us to put in another way what we have obtained by now.

Corollary 6. FSpec *is left adjoint to* Γ*, and the unit* $\mathrm{id}_{\mathbf{CRings}} \to \Gamma \circ \mathrm{FSpec}$ *of this adjunction is an isomorphism. Also,* FSpec *is full and faithful, and maps* **CRings** *onto a full subcategory of* **FGeom**.

In contrast even to the proof of this universal property in [22, V.3.5], we have never made essential use of points or stalks. Instead, the family of supports Θ from (24) has served well as a link between the two components of a formal geometry; in particular, it has turned out to be indispensable to put the locality condition in the point-free way of (29). Last but not least, note that neither positivity nor inequality were necessary to arrive at the universal property: these notions have occurred not even implicitly in Section 5.4.

6. Future work

In this article the paradigmatic example of an affine scheme, the Zariski topology of a commutative ring together with its structure sheaf, has given rise to a formal geometry. One next has to consider as a formal geometry the prime example of a non-affine scheme, the projective spectrum of a graded ring, of which task the topological part has already been performed [13]. The principal objective of a current project by Henri Lombardi and the author is to obtain, presumably within (a suitable variant of) the category of formal geometries, a sufficiently general notion of a "finitary" scheme that includes all these examples.

The concept of a (pre)sheaf on a formal topology still has to be extended to the more general variants of a formal topology present in the literature, first to the one with an arbitrary underlying set rather than a monoid. A further task is to provide the sheafification of a presheaf, probably as sections in its display space *(espace étalé)* yet to be conceived as a formal topology. This is work in progress by Erik Palmgren and the author. The reason why in this paper we have not followed either direction is that none of them is of any use in the present context, where the monoid structure comes with the ring multiplication, and the only presheaf under consideration actually is a sheaf.

Acknowledgements

The author is grateful to Jesper Carlström, Thierry Coquand, Giovanni Curi, Henri Lombardi, Erik Palmgren, and Giuseppe Rosolini for comments and suggestions. The anonymous referees' critique was most helpful for bringing this paper into its final form.

References

[1] P. Aczel, The type theoretic interpretation of constructive set theory, in: A. Macintyre, L. Pacholski, J. Paris (Eds.), Logic Colloquium'77, North-Holland, Amsterdam, 1978, pp. 55–66.
[2] P. Aczel, The type theoretic interpretation of constructive set theory: Choice principles, in: A.S. Troelstra, D. van Dalen (Eds.), The L.E.J. Brouwer Centenary Symposium, North-Holland, Amsterdam, 1982, pp. 1–40.
[3] P. Aczel, The type theoretic interpretation of constructive set theory: Inductive definitions, in: R.B. Marcus, et al. (Eds.), Logic, Methodology, and Philosophy of Science VII, North-Holland, Amsterdam, 1986, pp. 17–49.
[4] P. Aczel, Aspects of general topology in constructive set theory, Ann. Pure Appl. Logic 137 (2006) 3–29.
[5] P. Aczel, C. Fox, Separation properties in constructive topology, in: L. Crosilla, P. Schuster (Eds.), From Sets and Types to Topology and Analysis, in: Oxford Logic Guides, vol. 48, Oxford University Press, 2005, pp. 176–192.
[6] P. Aczel, M. Rathjen, Notes on constructive set theory, Institut Mittag–Leffler, Preprint 40, 2000/01.

[7] B. Banaschewski, The power of the ultrafilter theorem, J. London Math. Soc. 27 (1983) 193–202.
[8] B. Banaschewski, Radical ideals and coherent frames, Comment. Math. Univ. Carolin. 37 (1996) 349–370.
[9] D.S. Bridges, Prime and maximal ideals in the constructive theory of rings with inequality, Commun. Algebra 29 (2001) 2787–2803.
[10] D. Bridges, F. Richman, Varieties of Constructive Mathematics, in: London Math. Soc. Lect. Notes, vol. 97, Cambridge University Press, Cambridge, 1987.
[11] J. Carlström, Subsets, quotients and partial functions in Martin–Löf's type theory, in: H. Geuvers, F. Wiedijk (Eds.), Types for Proofs and Programs (Proc. Internat. Workshop TYPES 2002 (Berg en Dal, The Netherlands)), in: Lect. Notes Comput. Sci., vol. 2646, Springer, Berlin, Heidelberg, 2003.
[12] J. Carlström, Wheels—On division by zero, Math. Struct. Comput. Sci. 14 (2004) 143–184.
[13] T. Coquand, H. Lombardi, P. Schuster, The projective spectrum as a distributive lattice, Cah. Topol. Géom. Différ. Catég. 48 (2007) 220–228.
[14] T. Coquand, G. Sambin, J. Smith, S. Valentini, Inductive generation of formal topologies, Ann. Pure Appl. Logic 124 (2003) 71–106.
[15] N. Gambino, P. Schuster, Spatiality for formal topologies, Math. Struct. Comput. Sci. 17 (2007) 65–80.
[16] S. Gebellato, G. Sambin, The essence of continuity. G. Sambin, S. Gebellato, P. Martin–Löf, V. Capretta, The basic picture, Preprint, University of Padua, 2003. See also [37].
[17] S. Gebellato, G. Sambin, Pointfree continuity and convergence, In the same preprint as [16]. See also [37].
[18] M. Hakim, Topos Annelés et Schémas Relatifs, in: Ergeb. Math. Grenzgeb., vol. 64, Springer, Berlin, Heidelberg, 1972.
[19] R. Hartshorne, Algebraic Geometry, in: Grad. Texts Math., vol. 52, Springer, New York, 1977.
[20] P.T. Johnstone, Topos Theory, in: London Math. Soc. Monographs, vol. 10, Academic Press, London, 1977.
[21] P.T. Johnstone, Rings, fields, and spectra, J. Algebra 49 (1977) 238–260.
[22] P.T. Johnstone, Stone Spaces, in: Cambridge Studies Adv. Math., vol. 3, Cambridge University Press, Cambridge, 1982.
[23] A. Joyal, Le théorème de Chevalley–Tarski, Cahiers Topol. Géom. Différ. Catég. 16 (1975) 256–258.
[24] S. MacLane, I. Moerdijk, Sheaves in Geometry and Logic. A First Introduction to Topos Theory, Springer, New York, 1992.
[25] M.E. Maietti, About effective quotients in constructive type theory, in: Th. Altenkirch, W. Naraschewski, B. Reus (Eds.), Types for Proofs and Programs (Proc. Internat. Workshop TYPES '98 (Irsee, Germany)), in: Lect. Notes Comput. Sci., vol. 1657, Springer, Berlin, Heidelberg, 1999, pp. 164–178.
[26] M.E. Maietti, S. Valentini, Can you add power-sets to Martin–Löf intuitionistic set theory? Math. Logic Quart. 45 (1999) 521–532.
[27] P. Martin–Löf, Intuitionistic Type Theory. Notes by G. Sambin of a Series of Lectures Given in Padua, June 1980, in: Studies Proof Theory, vol. 1, Bibliopolis, Naples, 1984.
[28] P. Martin–Löf, G. Sambin, Generating positivity by coinduction, In the same preprint as [16] and in [37].
[29] R. Mines, F. Richman, W. Ruitenburg, A Course in Constructive Algebra, Springer, New York, 1987.
[30] S. Negri, Stone bases, alias the constructive content of Stone representation, in: A. Ursini, P. Aglianó (Eds.), Logic and Algebra (Proc. Conf. in Memoriam Roberto Magari (Pontignano, Siena, Italy, 1994)), Dekker, New York, 1996, pp. 617–636.
[31] H. Persson, An application of the constructive spectrum of a ring. H. Persson, type theory and the integrated logic of programs, Doctoral Thesis, Department of Computing Sciences, Chalmers and Gothenburg University, 1999.
[32] F. Richman, Nontrivial uses of trivial rings, Proc. Amer. Math. Soc. 103 (1988) 1012–1014.
[33] W. Ruitenburg, Intuitionistic algebra, theory and sheaf models, Doctoral Dissertation, University of Utrecht, 1982.
[34] G. Sambin, Intuitionistic formal spaces—a first communication, in: D. Skordev (Ed.), Mathematical Logic and its Applications (Conf. Proc. (Druzhba, Bulgaria, 1986)), Plenum, New York, London, 1987, pp. 187–204.
[35] G. Sambin, Some points in formal topology, Theoret. Comput. Sci. 305 (2003) 347–408.
[36] G. Sambin, The basic picture, a structure for topology, In the same preprint as [16]. See also [37].
[37] G. Sambin, The basic picture: A structural basis for constructive topology, in: Oxford Logic Guides, Oxford University Press (in preparation).
[38] G. Sambin, S. Gebellato, A preview of the basic picture: A new perspective on formal topology, in: T. Altenkirch, W. Naraschewski, B. Reus (Eds.), Types for Proofs and Programs (Proc. Internat. Workshop TYPES '98 (Kloster Irsee, Germany)), in: Lect. Notes Comput. Sci., vol. 1657, Springer, Berlin, 1999, pp. 194–207.
[39] G. Sambin, S. Valentini, Building up a toolbox for Martin–Löf's type theory: Subset theory, in: G. Sambin, J. Smith (Eds.), Twenty–five Years of Constructive Type Theory (Congress Proc. (Venice, Italy, 1995)), in: Oxford Logic Guides, vol. 36, Oxford University Press, Oxford, 1998, pp. 221–244.
[40] P. Schuster, Spectra and sheaves in formal topology, Habilitation Thesis, University of Munich, 2003.
[41] P. Schuster, Formal Zariski topology: Positivity and points, Ann. Pure Appl. Logic 137 (2006) 317–359.
[42] I. Sigstam, Formal spaces and their effective presentations, Arch. Math. Logic 34 (1995) 211–246.
[43] A.S. Troelstra, D. van Dalen, Constructivism in Mathematics. An Introduction, in: Studies Logic Found. Math., vol. 121, 123, North Holland, Amsterdam, 1988.
[44] G. Wraith, Generic Galois theory of local rings, in: M. Fourman, C. Mulvey, D. Scott (Eds.), Applications of Sheaves, in: Lect. Notes Math., vol. 753, Springer, Berlin, Heidelberg, 1977, pp. 739–767.